Semi-classical analysis and vanishing properties of solutions to quasilinear equations *

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Abstract

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ and $b$ a measurable nonnegative function in $\Omega$. We deal with the time compact support property for

$$u_t - \Delta u + b(x)|u|^{q-1}u = 0$$

for $p \geq 2$ and

$$u_t - \text{div}(|\nabla u|^{p-2}\nabla u) + b(x)|u|^{q-1}u = 0$$

with $m \geq 1$ where $0 \leq q < 1$. We give criteria associated to the first eigenvalue of some quasilinear Schrödinger operators in semi-classical limits. We also provide a lower bound for this eigenvalue.

1 Introduction

Let $\Omega$ be a regular bounded domain of $\mathbb{R}^N$ ($N \geq 1$) and $q \in [0, 1)$. We consider the weak solution of the degenerate parabolic equations subject to the Neumann boundary condition:

$$u_t - \Delta u + b(x)|u|^{q-1}u = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$\partial_{\nu}u = 0 \quad \text{on } \partial \Omega,$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega,$$

and more generally,

$$u_t - \text{div}(|\nabla u|^{p-2}\nabla u) + b(x)|u|^{q-1}u = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$\partial_{\nu}u = 0 \quad \text{on } \partial \Omega,$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega,$$

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with \( p \geq 2 \), or

\[
\begin{aligned}
  u_t - \Delta(u^m) + b(x)|u|^{q-1}u &= 0 \quad \text{in } \Omega \times (0, \infty), \\
  \partial_\nu u &= 0 \quad \text{on } \partial\Omega, \\
  u(x,0) &= u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]

with \( m \geq 1 \).

Many authors have already dealt with such equations giving a wide range of applications in physical mathematics. Now, our task is to describe a compact compact support property, in time.

**Definition.** A solution \( u \) satisfies the Time Compact Support property (for short TCS property) if there exists a time \( T \) such that for all \( t \geq T \) and all \( x \in \Omega \), \( u(x,t) = 0 \).

First, we study some simple cases for (1.1):

1) Suppose that there exists a real \( \gamma \) such as \( b(x) \geq \gamma > 0 \) a.e. in \( \Omega \). From the maximum principle, \( u(x,t) \leq (1 - \gamma(1-q)t)^{\frac{1}{1-q}} \) in \( \Omega \times (0, \infty) \). The nonlinear absorption is stronger than the diffusion and the TCS property holds.

2) We have a different feature if we assume that there exists a connected open set \( \omega \) such as \( b(x) = 0 \) a.e. in \( \omega \) (no absorption in \( \omega \)). Then usually, \( u \) has not the compact support property. Indeed, if we denote by \( \lambda(\omega) \) the first eigenvalue of \( -\Delta \) in \( W^{1,2}_0(\omega) \) and \( \zeta \) the first eigenfunction with \( \|\zeta\|_{L^\infty(\omega)} = 1 \) and \( \zeta \geq 0 \), then from the maximum principle, \( u(x,t) \geq \zeta(x) e^{-\lambda(\omega)t} \) for all \( x \in \omega \) and for all \( t \geq 0 \).

Up to some minor changes, the previous examples are also valid if \( u \) satisfies (1.2) and (1.3). The compact support property is related to \( \{ x : b(x) = 0 \} \) and the behaviour of the function \( b \) in a neighbourhood of this set.

### 2 The time compact support property

The starting idea was in the article of Kondratiev and Véron [7]. They established this property for (1.1) with the help of the following quantities

\[
\mu_n = \inf \left\{ \int_\Omega (|\nabla v|^2 + 2^n b(x)|v|^2) dx : v \in W^{1,2}(\Omega), \int_\Omega |v|^2 dx = 1 \right\},
\]

with \( n \) positive integer number. More precisely, up to a small change, they proved the following theorem.

**Theorem 2.1** Suppose that \( u \) is a solution of (1.1) and

\[
\sum_{n=0}^{+\infty} \frac{\ln \mu_n}{\mu_n} < +\infty,
\]

then there exists some \( T > 0 \) such that \( u(x,t) = 0 \) for \( (x,t) \in \Omega \times [T, +\infty) \).
We see that \( \mu_n \) are linked to well-known questions in the semi-classical limit of Schrödinger operator of the type 
\[-\Delta + 2^n b(.) \].

In [3], the authors give a first extension of this theorem by replacing the sequence \( 2^n \) by any decreasing sequence going to zero. For the sake of simplicity, we denote by \( \mu(\alpha) \) the lowest eigenvalue of the Neumann realization of the Schrödinger operator 
\[-\Delta + \alpha^{q-1} b(.) \] in \( W^{1,2}(\Omega) \), that is,

\[
\mu(\alpha) = \inf \left\{ \int_{\Omega} \left( |\nabla v|^2 + \alpha^{q-1} b(x)|v|^2 \right) dx : v \in W^{1,2}(\Omega), \int_{\Omega} |v|^2 dx = 1 \right\}.
\] (2.1)

They proved the following theorem [3, page 50].

**Theorem 2.2** Assume that \( (\alpha_n) \) is a decreasing sequence of positive numbers such that

\[
\sum_{n=1}^{+\infty} \frac{1}{\mu(\alpha_n)} (\ln(\mu(\alpha_n)) + \ln(\frac{\alpha_n}{\alpha_{n+1}}) + 1) < +\infty,
\] (2.2)

then any solution of (1.1) satisfies the TCS property.

The proof is based on an iterative method using the following lemma.

**Lemma 2.1** Suppose that \( b \geq 0 \) a.e. in \( \Omega \), \( 0 \leq q < 1 \) and \( u \) is a bounded weak solution of (1.1) such that \( \|u_0\|_{L^\infty(\Omega)} \leq \alpha \) for some \( \alpha > 0 \). Then

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \min \left( 1, C(\mu(\alpha))^N e^{-t\mu(\alpha)} \right) \|u_0\|_{L^\infty(\Omega)},
\] (2.3)

where \( C = C(\Omega) \) is a positive real number.

**Outline of the proof.** We use \( u \) as test-function and since \( u^{1-q} \geq \alpha^{1-q} \), we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \int_{\Omega} (|\nabla u|^2 + b\alpha^{q-1} u^2) dx \leq 0.
\]

The definition of \( \mu(\alpha) \) and Hölder’s inequality gives

\[
\|u(\cdot, s)\|_{L^2(\Omega)} \leq e^{-s\mu(\alpha)} \|u_0\|_{L^\infty(\Omega)},
\]

for all positive real number \( s \). The regularizing effects associated to this type of equation can be write under the following form [11, 12]:

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(1 + \frac{1}{t-s})^{N/4} \|u(\cdot, s)\|_{L^2(\Omega)},
\]

for all \( t > s \). Taking \( t - s = 1/\mu(\alpha) \) completes the proof of the lemma. \( \Box \)
Sketch of the proof of the theorem 2.2. \((\alpha_n)\) is a decreasing sequence which tends to zero. We shall construct an increasing sequence \((t_n)\) such that for all \(n\),

\[
\forall t \geq t_n, \|u(., t)\|_{L^\infty(\Omega)} \leq \alpha_n.
\]

If \(\lim_{n \to +\infty} t_n = T < +\infty\) then \(u\) satisfies the TCS property. To do this, we use an iterative method to find an upper bound for \(\sum t_{n+1} - t_n\) under the form of a convergent series. We set \(t_0 = 0\) and \(\alpha = \alpha_0 = \|u_0\|_{L^\infty(\Omega)}\). Applying Lemma 2.1 gives an upper bound for \(\|u(., t)\|_{L^\infty(\Omega)}\). \(t_1\) is defined by

\[
C(\mu(\alpha_0))^{N/4} e^{-(t_1-t_n)\mu(\alpha_0)} \alpha_0 = \alpha_1.
\]

A this point, we apply Lemma 2.1 but for time \(t \geq t_1\) with \(\alpha = \alpha_1\). Iterating this process provide us the formula

\[
C(\mu(\alpha_0))^{N/4} e^{-(t_{n+1}-t_n)\mu(\alpha_n)} \alpha_n = \alpha_{n+1}.
\]

So we obtain an upper bound for the series \(\sum t_{n+1} - t_n\). An analogous result can be proved for \((1.2)\). But before, we recall the regularizing effects for this type of equation [11, 12].

**Theorem 2.3** Let \(p > 1\). Suppose that \(u\) is a weak solution of

\[
\begin{align*}
    u_t - \text{div}(\nabla u |p-2\nabla u) + B(x, t, u) &= 0 \quad \text{in } \Omega \times (0, \infty), \\
    \partial_\nu u &= 0 \quad \text{on } \partial \Omega, \\
    u(x, 0) &= u_0(x) \in L^r(\Omega),
\end{align*}
\]

where \(B\) is a Caratheodory functions which satisfies \(B(x, t, \rho)\rho \geq 0\) a.e. in \(\Omega \times (0, \infty)\). If \(r \geq 1, r > N(2/p - 1)\) then

\[
\|u(., t)\|_{L^\infty(\Omega)} \leq C(1 + \frac{1}{r})^\delta(r) \|u(., 0)\|_{L^\sigma(r)},
\]

with \(C = C(\Omega, p), \delta(r) = \frac{N}{r p + N(p-2)}\) and \(\sigma(r) = \frac{r p}{r p + N(p-2)}\).

In a similar way, we introduce

\[
\mu(\alpha, p) = \inf \left\{ \int_\Omega (|\nabla v|^p + \alpha^{r-(p-1)} b(x)|v|^p) \, dx : v \in W^{1, p}(\Omega), \int_\Omega |v|^p \, dx = 1 \right\}.
\]

Here \(\mu(\alpha, p)\) is the first eigenvalue in \(W^{1, p}(\Omega)\) for the Neumann boundary condition

\[
u \mapsto -\Delta_\nu u + \alpha^{r-(p-1)} b(\nu)u^{p-1}.
\]

The theorem states as follows [1]:

**Theorem 2.4** Let \(0 < q < 1, p > 2\) and assume that there exist two sequences of positive real numbers \((\alpha_n)\) and \((\gamma_n)\) such that \((\alpha_n)\) is decreasing and

\[
\sum_{n=0}^{\infty} \frac{\alpha_n^{p-2} \mu(\alpha_n, p) \sigma(\gamma_n)}{\alpha_n^{p-2} \mu(\alpha_n, p) \sigma(\gamma_n)} < +\infty. \quad (2.4)
\]

Then any solution of \((1.2)\) with initial bounded data satisfies the TCS property.
Consequently, if \( r_n = \ln \mu(\alpha_n, p) \), we have the following statement.

**Corollary 2.1** Under the same assumptions on \( q \) and \( p \), if there exists a decreasing sequence of positive real numbers \( (\alpha_n) \) such that

\[
\sum_{n=0}^{\infty} \frac{(\ln \mu(\alpha_n, p))^{p-1}}{\alpha_{n+1}^{p-1}} p < +\infty,
\]

then any solution of (1.2) satisfies the TCS property.

Theorem 2.4 comes from the following lemma.

**Lemma 2.2** Suppose there exists a measurable function \( u \) in \( \Omega \times \mathbb{R}^+ \) which satisfies weakly (1.2) with \( \|u_0\|_{L^\infty(\Omega)} \leq \alpha \) for some \( \alpha > 0 \). Then

\[
\|u(., t)\|_{L^\infty(\Omega)} \leq \left( \frac{1}{\|u(., 0)\|_{L^{2-p}(\Omega)}^{2-p} + C_1 \mu(\alpha, p) t} \right)^{\frac{p}{2-p}},
\]

where \( C_1 = C_1(\Omega, r, p) \) is a positive real constant and there exist two positive real numbers \( C = C(\Omega, p) \) and \( C_2 = C_2(r, p) \) such that

\[
\|u(., t)\|_{L^{\infty}(\Omega)} \leq \min \left( C(1 + \frac{2}{t})^{\delta(r)} \left( \frac{1}{\|u(., 0)\|_{L^{2-p}(\Omega)}^{2-p} + C_2 \mu(\alpha, p) t} \right)^{\frac{\sigma(r)}{2-p}}, 1 \right),
\]

with \( \delta(r) = \frac{N}{rp + N(p-2)} \) and \( \sigma(r) = \frac{rp}{rp + N(p-2)} \).

**Idea in the proofs.** The principle to prove them remains true. It is a bit more complicated because the term \( u_t \) is not homogenous with \( u^{p-1} \) but it follows exactly the Kondratiev-Vron method as shown in the proof of Theorem 2.2. The main differences are technical. Instead of using \( u \) as test-function, we use \( u| u|^{\alpha-1} \) at each step of the iteration. An estimate of the asymptotic behaviour when \( r \to +\infty \) for the constant \( C_2 = C_2(r, p) \) is needed. The proof of the theorem ends with sharp upper bounds for the series \( \sum t_{n+1} - t_n \). \( \Box \)

Now, let us talk about equation 1.3. Formally, replacing \( p-1 \) by \( m \) give the same results [11, 12]:

**Theorem 2.5** Let \( m > 0 \) and \( u \) be a weak solution of

\[
u_t - \Delta(u^m) + B(x, t, u) = 0 \quad \text{in} \ \Omega \times (0, \infty),
\]

\[
\partial_n u = 0 \quad \text{on} \ \partial\Omega,
\]

\[
u(x, 0) = u_0(x) \in L^r(\Omega),
\]

where \( B \) is a Caratheodory function satisfying \( B(x, t, \rho) \rho \geq 0 \) a.e. in \( \Omega \times (0, \infty) \). If \( r \geq 1 \) and \( r > N(1-m)/2 \), then

\[
\|u(., t)\|_{L^{\infty}(\Omega)} \leq C(1 + \frac{1}{t})^{\delta(r)} \|u(., 0)\|_{L^r(\Omega)}^{\sigma(r)} ,
\]

with \( C = C(\Omega, m) \), \( \delta(r) = \frac{N}{2r + N(m-1)} \) and \( \sigma(r) = \frac{2r}{2r + N(m-1)} \).
We set quantities adapted to the problem
\[
\mu'(\alpha, m) = \inf \left\{ \int_\Omega \left( \|\nabla v\|^2 + \alpha^{r-m} b(x)|v|^2 \right) dx : v \in W^{1,2}(\Omega), \int_\Omega |v|^2 dx = 1 \right\}.
\]

Thus,

**Theorem 2.6 ([1])** Let \(0 \leq q < 1\), \(m > 1\) and assume that there exists two sequences of positive real numbers \((\alpha_n)\) and \((r_n)\) such that \((\alpha_n)\) is decreasing and

\[
\sum_{n=0}^{\infty} \frac{r_n^m}{\alpha_{n+1}^{m-1}} \mu'(\alpha_n, m)^{\sigma(r_n)} < +\infty.
\]

(2.7)

Then any solution of (1.3) with initial bounded data satisfies the TCS property.

With \(r_n = \ln \mu'(\alpha_n, m)\), we deduce the following statement.

**Corollary 2.2** Under the above assumptions on \(q\) and \(m\), if there exists a decreasing sequence of positive real numbers \((\alpha_n)\) such that

\[
\sum_{n=0}^{\infty} \frac{(\ln \mu'(\alpha_n, m))^m}{\alpha_{n+1}^{m-1}} \mu'(\alpha_n, m)^{\sigma(r_n)} < +\infty,
\]

then any solution of (1.3) satisfies the TCS property.

The proof of Theorem 2.6 also comes from the following lemma.

**Lemma 2.3** We suppose there exists a measurable function \(u\in \Omega \times \mathbb{R}^+\) which satisfies weakly (1.3) with \(\|u(\cdot,0)\|_{L^\infty(\Omega)} \leq \alpha\) for some \(\alpha > 0\). Then

\[
\|u(\cdot,t)\|_{L^r(\Omega)} \leq \left( \frac{1}{\|u(\cdot,0)\|^m_{L^r(\Omega)} + C_1 \mu'(\alpha, m) t} \right)^{1/(m-1)},
\]

(2.8)

with \(C_1 = C_1(\Omega, r, m)\) and there exist two positive real numbers \(C = C(\Omega, m)\) and \(C_2 = C_2(\Omega, m)\) such that

\[
\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \min \left( C(1+\frac{2}{m})^{\delta(r)} \left( \frac{1}{\|u(\cdot,0)\|^m_{L^\infty(\Omega)} + C_2 \mu'(\alpha, m) t} \right)^{\sigma(r)/(m-1)}, 1 \right),
\]

where \(\delta(r)\) and \(\sigma(r)\) are defined in Theorem 2.5.

The assumptions in Theorem 2.2 and Corollaries 2.1, 2.2 admit a simpler form. A comparison between series and integral gives the following theorem.

**Theorem 2.7** (Integral criterion [3, 1]) Let \(0 \leq q < 1\). 1) If \(p \geq 2\) and

\[
\int_0^1 \frac{(\ln \mu(t,p))^p}{t^{p-1}} \mu(t,p) dt < +\infty,
\]
then all solutions of (1.2) satisfy the TCS property.

2) If \( m \geq 1 \) and
\[
\int_0^1 \frac{(\ln \mu'(t, m))^m}{t^m \mu'(t, m)} \, dt < +\infty,
\]
then all solutions of (1.3) satisfy the TCS property.

We remark that \( \mu(t) = \mu(t, 2) \) and that (1.1) is a particular case of (1.2) for \( p = 2 \) and (1.3) for \( m = 1 \). The proof is first establish for \( p = 2 \) [3, page 51] and then for \( p > 2 \) and \( m > 1 \) [1]. What is remarkable is that this criterion has a same simple form in all cases.

For applications, \( \mu(t, p) \) and \( \mu'(t, m) \) have to be linked directly to the function \( b \). We recall that \( \mu(\alpha, p) \) is the first eigenvalue in \( W^{1,p}(\Omega) \) for the Neumann boundary condition of \( u \mapsto -\Delta_p u + \alpha q - (p-1)b(\cdot)u^{p-1} \).

The aim of semi-classical analysis is to describe the behavior of the spectrum of the operator \( u \mapsto -\Delta_p u + h^{-p}V(\cdot)u^{p-1} \) in particular \( \lambda_1(h) \) the lowest eigenvalue. \( V \) is a function which holds in our case
\[
V \in L^\infty(\Omega), \quad \text{ess inf } \Omega V = 0 \quad \text{and} \quad \int_\Omega V(x) \, dx > 0. 
\] (2.9)

We denote by \( \gamma \) a positive number which satisfies:
\[
\gamma = \begin{cases} 
\frac{N}{p} & \text{for } 1 < p < N, \\
\in (1, +\infty) & \text{for } p = N, \\
1 & \text{for } p > N,
\end{cases} 
\] (2.10)

**Corollary 2.3** If (2.9) holds then for \( h \) small enough,
\[
\lambda_1(h)(\text{meas}\{x : V(x) \leq h^p \lambda_1(h)\})^{1/\gamma} \geq C, 
\] (2.11)

where \( C = C(p, N, \gamma, \Omega, V) \) is a positive constant.

\( \mu(t, p) \) can be written as \( \mu(t, p) = \lambda_1(t^{\frac{(p-1)-q}{p}}) \) which after a change of variables gives
\[
\int_0^1 \frac{(\ln \mu(t, p))^{p-1}}{t^{p-1} \mu(t, p)} \, dt = \int_0^1 \frac{(\ln \lambda_1(h))^{p-1}}{h^{\frac{(p-1)+q}{p-1}(\frac{p}{p-1})}} \, dh. 
\]

If we have an estimate of the type
\[
\lambda_1(h) \geq C \frac{1}{h^{\theta}},
\]
where \( C \) and \( \theta \) are two positive real numbers, then the integral criterion holds for \( p > 2 \) provided
\[
\theta > \frac{p(p-2)}{p - (1 + q)}. 
\] (2.12)

Similar expressions can be found for \( p = 2 \) and \( m > 1 \). Finally, we obtain next theorem.
Theorem 2.8 (1/b criterion [3, 1]) Let $0 \leq q < 1$ and $b$ be a bounded measurable function such that
\[
\inf_{\Omega} b = 0 \quad \text{and} \quad \int_{\Omega} b(x) \, dx > 0.
\]
1) If $p = 2$ and $\ln(1/b) \in L^s(\Omega)$ for some $s > N/2$ then equation (1.1) satisfies the TCS property.
2) If $p > 2$ and $(1/b)^s \in L^1(\Omega)$ for some $s$ with
\[
s > \begin{cases} 
\frac{p-2}{p-2} \frac{N}{p} & \text{for } p \leq N, \\
\frac{m-1}{1-q} & \text{for } p > N,
\end{cases}
\]
then equation (1.2) satisfies the TCS property.
3) If $m > 1$ and $(1/b)^s \in L^1(\Omega)$ for some $s$ with
\[
s > \begin{cases} 
\frac{m-1}{1-q} \frac{N}{2} & \text{for } N \geq 2, \\
\frac{m-1}{1-q} & \text{for } N = 1,
\end{cases}
\]
then equation (1.3) satisfies the TCS property.

Outline of the proof. The three cases are based on Marcinkiewicz type inequalities. For 1)
\[
\operatorname{meas} \left\{ x \in \Omega : \ln \frac{1}{b(x)} \geq \frac{1}{h^2 \lambda_1(h)} \right\} \leq \frac{1}{(\ln \frac{1}{h^2 \lambda_1(h)})^s} \int_{\Omega} \left( \frac{1}{b(x)} \right)^s \, dx,
\]
and for 2)
\[
\operatorname{meas} \left\{ x : \frac{1}{b(x)} \geq \frac{1}{h^p \lambda_1(h)} \right\} \leq (h^p \lambda_1(h))^s \int_{\Omega} \left( \frac{1}{b(x)} \right)^s \, dx.
\]
The proof ends with estimates such as (2.12) and some technical arguments. □

Remark 2.1 In the case where $p = 2$ and $N \leq 2$, estimate (2.11) is not enough sharp so we use the formula of Lieb and Thirring. See [3] for details.

Now we apply the previous theorem to the radial functions.

Corollary 2.4 Suppose that $0 \in \Omega$. 1) If $b(x) = \exp\left( -\frac{1}{\|x\|^\beta} \right)$ with $\beta < 2$ then any solution of (1.1) satisfies the TCS property.
2) If $b(x) = \|x\|^\beta$ with $p \leq N$ and $\beta < p(1-q)/(p-2)$ then any solution of (1.2) satisfies the TCS property.
One has the same conclusion if $p > N$ and $\beta < N(1-q)/(p-2)$.
3) If $b(x) = \|x\|^\beta$ with $N \geq 2$ and $\beta < 2(1-q)/(m-1)$ then any solution of (1.3) satisfies the TCS property.
One has the same conclusion if $N = 1$ and $\beta < (1-q)/(m-1)$. 

3 A lower bound for the first eigenvalue

This section is dedicated to estimating the first eigenvalue, in $W^{1,p}(\Omega)$, of the operator $u \mapsto -\Delta_p u + h^{-p}V(.)u^{p-1}$. We have seen that a lower bound is fundamental for applications. First, we introduce a sequence of definitions. We consider a non-empty connected open subset $\Omega \subset \mathbb{R}^N$ and a measurable function $V$ defined in $\Omega$. We set

$$W^{1,p,V}(\Omega) = \{ \psi \in W^{1,p}(\Omega) : V(x)|\psi|^p \in L^1(\Omega) \}.$$  

If $W^{1,p,V}(\Omega) \neq \{0\}$ and $\psi \in W^{1,p,V}(\Omega)$, we set

$$F_V(\psi) = \int_{\Omega} |\nabla \psi|^p + V(x)|\psi|^p \, dx, \quad (3.1)$$

and define

$$\lambda_1 = \inf \left\{ F_V(\psi) : \psi \in W^{1,p,V}(\Omega), \int_{\Omega} |\psi|^p \, dx = 1 \right\}, \quad (3.2)$$

and for $h > 0$,

$$\lambda_1(h) = \inf \left\{ F_{h^{-p}V}(\psi) : \psi \in W^{1,p,V}(\Omega), \int_{\Omega} |\psi|^p \, dx = 1 \right\}, \quad (3.3)$$

Thus $\lambda_1(h)$ is the first eigenvalue of the operator

$$u \mapsto -\Delta_p u + h^{-p}V(.)|u|^{p-2}u.$$ \quad (3.4)

in $W^{1,p,V}(\Omega)$ with Neumann boundary condition if the infimum is achieved by a regular enough element of $W^{1,p,V}(\Omega)$ and $\partial \Omega \in C^1$.

We start with a simple result which enlights our arguments. On the contrary to the linear case ($p = 2$), our proof is not based on the theory of pseudodifferential operators but on the continuous injections of $W^{1,p}(\Omega)$ into the $L^s$ spaces for suitable $s$.

**Theorem 3.1** Suppose $N > p > 1$. Then either $\lambda_1 = -\infty$ or

$$\left( \int_{V(x) \leq \lambda_1} (\lambda_1 - V(x))^{N/p} \, dx \right)^{p/N} \geq C(p,N), \quad (3.5)$$

where $C = C(p,N) > 0$ is the positive constant of the Sobolev inequality. In addition, if there exists a minimizer in $W^{1,p,V}(\mathbb{R}^N)$,

$$\left( \int_{V(x) < \lambda_1} (\lambda_1 - V(x))^{N/p} \, dx \right)^{p/N} \geq C(p,N). \quad (3.6)$$
Proof. Let $\psi$ be in $W^{1,p,V}(\mathbb{R}^N)$ with $\|\psi\|_{L^p(\mathbb{R}^N)} = 1$ then
\[
\int_{\mathbb{R}^N} |\nabla \psi|^p \, dx + \int_{\mathbb{R}^N} V(x)|\psi|^p \, dx = F_V(\psi) = F_V(\psi) \int_{\mathbb{R}^N} |\psi|^p \, dx.
\]
The integral with $V$ is split in two parts, that is, $\mathbb{R}^N = \{ x : V(x) < F_V(\psi) \} \cup \{ x : V(x) \geq F_V(\psi) \}$. Therefore,
\[
\int_{\mathbb{R}^N} |\nabla \psi|^p \, dx \leq \int_{V(x)<F_V(\psi)} (F_V(\psi) - V(x)) \rho(x) \, dx. \tag{3.7}
\]
Hölder’s inequality leads to
\[
\int_{\mathbb{R}^N} |\nabla \psi|^p \, dx \leq \left( \int_{V(x)<F_V(\psi)} (F_V(\psi) - V(x))^N/p \, dx \right)^{p/N} \left( \int_{\mathbb{R}^N} \rho(x)^p \, dx \right)^{1 - \frac{p}{N}}. \tag{3.8}
\]
since $\{ x : V(x) < F_V(\psi) \} \subset \mathbb{R}^N$. Non zero constants do not belong to $W^{1,p,V}(\mathbb{R}^N)$ and so all functions $\psi$ satisfy $\int_{\mathbb{R}^N} |\nabla \psi|^p \, dx > 0$. We can apply Sobolev inequality. The Beppo-Levi theorem completes the proof. □

Remark 3.1 If $\Omega$ is any open domain of $\mathbb{R}^N$, we define
\[
W^{1,p,V}_0(\Omega) = \{ \psi \in W^{1,p}_0(\Omega) : V(x)|\psi|^p \in L^1(\Omega) \},
\]
and if $W^{1,p,V}_0(\Omega) \neq \{0\}$,
\[
\tilde{\lambda}_1 = \inf \{ F_V(\psi) : \psi \in W^{1,p,V}_0(\Omega), \int_{\Omega} |\psi|^p \, dx = 1 \},
\]
then the estimates in Theorem 3.1 hold for $\tilde{\lambda}_1$.

When $\Omega$ is a $C^1$ bounded domain of $\mathbb{R}^N$ and $V$ is a measurable function such that
\[
V \in L^\infty(\Omega), \quad \text{ess inf}_\Omega V = 0 \quad \text{and} \quad \int_{\Omega} V(x) \, dx > 0, \tag{3.9}
\]
we set $u_h$ the first eigenfunction related to the first eigenvalue $\lambda_1(h)$.

Recall that $\gamma$ is a positive number which satisfies
\[
\gamma \begin{cases} = \frac{N}{p} & \text{for } 1 < p < N, \\ \in (1, +\infty) & \text{for } p = N, \\ = 1 & \text{for } p > N, \end{cases} \tag{3.10}
\]
with $\gamma^{\frac{1}{\gamma-1}} = +\infty$ if $\gamma = 1$. This $\gamma$ is such that $W^{1,p}$ imbeds $L^q(\Omega)$ continuously with $q = p \frac{\gamma}{\gamma-1}$.

Theorem 3.2 Assume that (3.9) holds. Then for $h$ small enough,
\[
\left( \int_{V(x)<h^{\gamma_1}(h)} (\lambda_1(h) - \frac{V(x)}{h^p})^{\gamma} \, dx \right)^{1/\gamma} \geq C,
\]
where $C = C(p, N, \gamma, \Omega, V)$ is a positive real constant.
Proof. We start with (3.8) because the beginning is similar. Replacing $\mathbb{R}^N$, $\psi$ and $V$ by $\Omega$, $u_h$ and $V_h$ the Hölder’s inequality gives

$$\int_{\Omega} |\nabla u_h|^p \, dx \leq \left( \int_{V(x) < h^p \lambda_1(h)} (\lambda_1(h) - \frac{V(x)}{h^p})^\gamma \, dx \right)^{1/\gamma} \left( \int_{\Omega} |u_h|^q \, dx \right)^{p/q},$$

where $q = \frac{p}{\gamma}$. Thus, by the imbeddings,

$$\left( \int_{V(x) < h^p \lambda_1(h)} (\lambda_1(h) - \frac{V(x)}{h^p})^\gamma \, dx \right)^{1/\gamma} \geq C \frac{\|\nabla u_h\|_{L^p(\Omega)}}{1 + \|\nabla u_h\|_{L^p(\Omega)}},$$

with $C = C(p, N, \Omega, \gamma)$ a positive real number. The main idea is to prove that

$$\liminf_{h \to 0} \|\nabla u_h\|_{L^p(\Omega)} > 0.$$

Suppose that there exists a sequence $\{h_n\}$ of positive real numbers which goes to zero such that

$$\lim_{n \to +\infty} \|\nabla u_{h_n}\|_{L^p(\Omega)} = 0.$$

Hence $(u_{h_n})$ is bounded in $W^{1,p}(\Omega)$, so there exists a function $u_0$ in $W^{1,p}(\Omega)$ such that, up to a subsequence, $u_{h_n} \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega)$. Obviously, $\|\nabla u_0\|_{L^p(\Omega)} = 0$. Therefore, $u_0 = C$ where $C$ is a real. Thanks to the Rellich-Kondrachov theorem, up to a subsequence, $u_{h_n} \to C$ strongly in $L^p(\Omega)$ so

$$C = \left( \frac{1}{\text{meas}(\Omega)} \right)^{1/p}.$$

We deduce that $\lim_{n \to +\infty} h_n^p \lambda_1(h_n) = \frac{\int V(x) \, dx}{\text{meas}(\Omega)}$. But from lemma 3.2 in [3], $\lim_{h \to 0} h^p \lambda_1(h) = 0$ which leads to a contradiction. □

A simpler form is provided in the following corollary.

Corollary 3.1 If (3.9) holds then for $h$ small enough,

$$\lambda_1(h) (\text{meas} \{x : V(x) < h^p \lambda_1(h)\})^\gamma \geq C,$$

where $C = C(p, N, \gamma, \Omega, V)$.

We end this section by quoting a theorem. For $\Omega$ a domain of $\mathbb{R}^N$ bounded or not, regular or not and $V$ a measurable function defined on $\Omega$ such that $W^{1,p,V}(\Omega) \neq \{0\}$, we define a well for a measurable function $V$ [1].

Definition. We say that $V$ has a well in $U$ if $U$ is a $C^1$ bounded, connected, non-empty open set of $\Omega$ and if there exists $\psi_0 \in W^{1,p,V}(\Omega)$ with $\|\psi_0\|_{L^p(\Omega)} = 1$ such that $\int_{\Omega \setminus U} V(x) |\psi_0|^p \, dx < a = \text{essinf}_{\Omega \setminus U} V$ with $\text{meas}(\Omega \setminus U) > 0$.

The term of well generalizes the definition in [8].

Theorem 3.3 ([3]) If $V$ has a well in $U$, for $h$ small enough,

$$\left( \int_{V(x) \leq h^p \lambda_1(h)} (\lambda_1(h) - h^{-p} V(x))^\gamma \, dx \right)^{1/\gamma} \geq C,$$
where $C$ is a positive constant which does not depend on $h$.

In addition, if there exists a minimizer in $W^{1,p, V}(\Omega)$,

$$
\left(\int_{V(x) < h^{\lambda_1}(h)} (\lambda_1(h) - h^{-p}V(x))^\gamma \, dx\right)^{1/\gamma} \geq C.
$$

The proof is technical but some arguments have already been used for Theorem 3.2.

4 Summary and open questions

For the sake of completeness, we quote another theorem of.

Theorem 4.1 ([3]) Suppose that $b$ is a continuous and nonnegative function defined in $\Omega$ which satisfies for some $x_0 \in \Omega$

$$
\lim_{r \to 0} r^2 \ln(1/\|b\|_{L^\infty(B_r(x_0))}) = \infty.
$$

If $u$ is a weak solution of (1.1) then $u$ does not satisfies the TCS property.

Up to now, we have the following:

<table>
<thead>
<tr>
<th>$p = 2$</th>
<th>$p &gt; 2$</th>
<th>$m &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integral criterion</td>
<td>$\int_0^1 \ln(\mu(t)) \frac{dt}{\mu(t)} &lt; \infty$</td>
<td>$\int_0^1 (\ln(\mu(t,p)))^{p-1} \frac{dt}{\mu(t,p)} &lt; \infty$</td>
</tr>
<tr>
<td>$1/b$ criterion with</td>
<td>$\ln(1/b) \in L^s$</td>
<td>$1/b \in L^s$</td>
</tr>
<tr>
<td>$s &gt; \frac{N}{2}$</td>
<td>$s &gt; \frac{p-2}{1-q} N$, $N \geq p$</td>
<td>$s &gt; \frac{p-2}{1-q} N$, $N \geq 2$</td>
</tr>
<tr>
<td>Radial case for $\beta \geq 0$ and</td>
<td>$\exp(-1/|x|^\beta)$</td>
<td>$\frac{|x|^\beta}{p(1-q)}$, $N \geq p$</td>
</tr>
<tr>
<td>$\beta &lt; 2$</td>
<td>$\beta &lt; \frac{N(1-q)}{p-2}$, $N &lt; p$</td>
<td>$\beta &lt; \frac{1-q}{m-1}$, $N = 1$</td>
</tr>
<tr>
<td>Converse</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Non TCS property for</td>
<td>$\exp(-1/|x|^\beta)$</td>
<td>:</td>
</tr>
<tr>
<td>$\beta &gt; 2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Open questions

1. What happens for $p = 2$ and $\beta = 2$? It does not seem within sight.
2. We have no genuine converse for $p > 2$ and $m > 1$. A converse has been found for $p = 2$ because $L^2(\Omega)$ has an inner product. More precisely, for $p > 2$, $\int_\Omega u^{p-1}v\,dx \neq \int_\Omega v^{p-1}u\,dx$ in general. We search for another test-functions (see [3] for details).

3. When $p > 2$, we have a good generalization of the Cwikel, Lieb and Rosenblyum formula, that is, for large dimension ($N > p$). The estimate for $N \leq p$ is far from the optimum. When $p = 2$, the Lieb and Thirring formula works well. We hope that we will find an equivalent.

4. In [7], they also deal with second order elliptic equations with a strong absorption, i.e., $u_{tt} + \Delta u - a(x)u^q = 0$. Heuristically speaking, changing $\mu(\alpha)$ into $\sqrt{\mu(\alpha)}$ gives a sufficient condition for the TCS property. We are working on this type of equation when $a$ depends also on $t$.

5. More generally, the following problem $\Delta_p u - a(x)u^{p-1} = 0$ in an outside domain is difficult to handle. On $\mathbb{R}^N$ minus a ball, a similar technique may be possible.

References


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